L 6 Sketches.

We have seen quite a number of categories that can be axiomatized by Lawvere theories or by their infinitary cousins.

Grp, Mon, Ring, Set, Set^C, Ab, R-Mod ...

these categories are called varieties and we discussed them in the third lecture of this course.

- The problem of non-empty sets $Set_{>0}$

Consider the category of non-empty sets. It cannot be a variety because we have seen that varieties are always cocomplete.

In this lecture we will present a gadget that allows to axiomatize a very broad class of categories, including varieties.

Def A sketch $\mathcal{S} = (S, \mathcal{L}, \mathcal{C})$ is the specification of the following data

- S a category (small). (large) (locally small).
- $\mathcal{L}$ is a set of cones on functors defined on small categories
- $\mathcal{C}$ is a set of cocones on functors defined on small categories.

<u>Example</u> Every category with finite products $\mathcal{C}$ has a sketch structure

$$\mathcal{S} = (\mathcal{C}, \text{product diagrams}, \emptyset).$$

By "product diagrams" we intend all the spans of the form

$$\begin{tikzcd} & A \times B \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

<u>Def</u> A sketch is

- normal if all its cones and cocones are of limit/colimit type.
- limit if $\mathcal{C}$ is empty.
- colimit if $\mathcal{L}$ is empty.
- mixed "otherwise".

It follows that the sketch associated to a category with finite products is a normal limit sketch.

<u>Def</u> A morphism of sketches $f : \mathcal{S} \longrightarrow \mathcal{Y}$ is a functor preserving the structure this gives us the 2-category of sketches $\underline{\mathrm{Skt}}$.

<u>Example</u> The category of Sets has a structure of "illegitimate" sketch were we put all limit and colimit cones

$$(\mathrm{Set}, \text{all}, \text{all}).$$

Def "the category of models of a sketch" $\mathrm{Mod}(\mathcal{S})$ is

$$\mathrm{Skt}(\mathcal{S}, \mathrm{Set}).$$

Example (Recovering universal algebra). If $\mathcal{C}$ is a small category with finite products, then its associated sketch $\mathcal{S}_C = (\mathcal{C}, \underset{\text{products}}{\text{finite}}, \emptyset)$ has the same models

$$\mathrm{Mod}(\mathcal{S}_C) = \mathrm{Skt}(\mathcal{S}_C, \mathrm{Set}) = \mathrm{Prod}_{\mathrm{fin}}(\mathcal{C}, \mathrm{Set}) = \mathrm{Mod}(\mathcal{C}).$$

A similar argument would work for infinitary variations of the theory.

Rem Of course, $\mathrm{Mod}(\mathcal{S})$, for $\mathcal{S}$ any sketch, is a full subcategory of $\mathrm{Set}^{\mathcal{S}}$

$$\mathrm{Mod}(\mathcal{S}) \overset{i}{\hookrightarrow} \mathrm{Set}^{\mathcal{S}}$$

but in full generality i does not preserve any limit / colimit (even when $\mathcal{S}$ is normal).

- If $\mathcal{C}$ is empty, it preserves limits.
- If $\mathcal{L}$ is empty it preserves colimits.
- As it happens for algebraic structures, i preserves all the colimits that commute with the limits in $\mathcal{L}$, and all the limits that commute with $\mathcal{C}$.

So, what are sketches useful for?

<u>Example</u> $\boxed{\mathbf{Set}_{>0}}$

- Let $\mathcal{S}$ be the category with two objects a, b and a unique nontrivial morphism
$$a \longrightarrow b$$
- Let $\mathcal{L}$ consist of the unique diagram which is empty and b as a cone for it.
- Let $\mathcal{C}$ consist of the cocone

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
b \arrow[dr, equal] & & b \arrow[dl, equal] \\
 & b &
\end{tikzcd}$$

diagram

cocone.

A model of $\mathcal{S}$ is the choice of two sets
$$A \xrightarrow{f} B$$
and a function between them such that

(a) the condition on the limit part forces B to be terminal

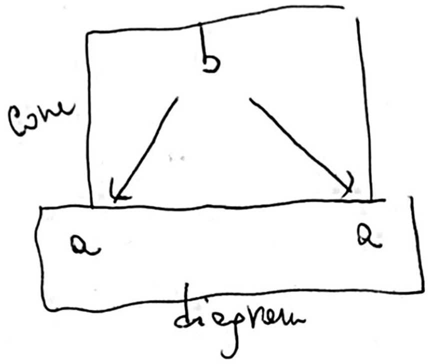
(b) the condition on the colimit part forces B to be an epimorphism

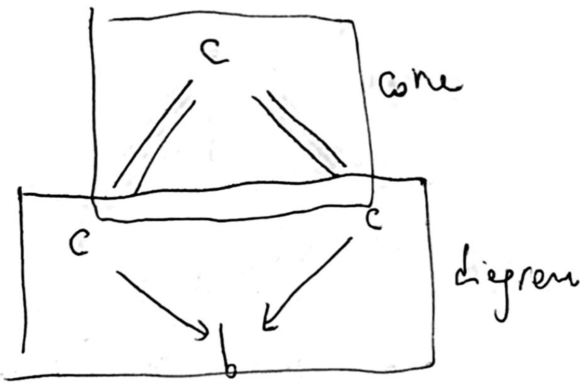
$$\Longrightarrow \quad \mathrm{Mod}(\mathcal{S}) = \mathbf{Set}_{>0}.$$

Example Poset. $\mathcal{J}$ has tree objes and arrows like below

$$\begin{tikzcd} c \arrow[r] & b \arrow[r, shift left] \arrow[r, shift right] & a \end{tikzcd}$$

Now in $\mathcal{L}$ we add





So, a model of $\mathcal{J}$ is a set A, the

$$\begin{tikzcd} R \arrow[r, hook] & A \times A \arrow[r, shift left] \arrow[r, shift right] & A \end{tikzcd}$$

data on $A \times A$ is redundant and a subset of the product R.

to force R to be a reflexive relation we need to add an object

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[dr, "h"'] & c \arrow[d, "g"] \\ & b \end{tikzcd}$$

and force $\boxed{h}$ to be monic.

Then a cone forcing $\boxed{h}$ to be the equalizer of the two projections

You can do as many examples as you want.

try with

• Categories	• Graphs
• Setoids	• Groupoids
• Fields	
• the terminal category	
• $Set_{\geq \lambda}$	
• and so on..	

<u>Rem</u> (Morita theory) Different sketches can have the same models. For example Set can be presented by the following sketches

$\mathcal{S}_1$	$\mathcal{S}_2$
$(\bullet\ ,\ \emptyset\ ,\ \emptyset)$	$(a \to b\ ,\ $ "b is terminal" $,\ \emptyset)$.

In this case the functor

$$\mathcal{S}_1 \overset{i}{\hookrightarrow} \mathcal{S}_2 \quad \text{mapping } \bullet \text{ to } a$$

is a "morita equivalence" in the sense that the two sketch has the same models and i^* induces the equivalence.

<u>Rem</u> (<u>Morita theory bis</u>) Let T be a finitary monad on Set. Then the inclusion

$$Kl_\omega(T)^{op} \xhookrightarrow{\ i\ } Kl(T)^{op}$$

is a Morita equivalence of sketches where on both side we put the "correct" sketch structure

$$\left(\begin{array}{c}\text{finite}\\ \text{products}\end{array}\right) \Bigg| \left(\text{All products}\right)$$

<u>Rem</u> $\mathrm{Mod}(\mathcal{S})$ might <u>not</u> be (ω) complete.

<u>Rem</u> the Yoneda embedding might not factor! So no trivial models!!

$$\begin{tikzcd}
 & \mathcal{S}^{op} \arrow[d, "よ"] \arrow[dl, dashed, "\text{NO!}"'] \\
\mathrm{Mod}(\mathcal{S}) \arrow[r, hook] & \mathrm{Set}^{\mathcal{S}}
\end{tikzcd}$$

<u>Example</u> $\mathcal{S} = (\bullet, \emptyset,$ "force $\bullet$ to be initial"$)$.

So everything seems sketchable, but is it so?!

The tautological sketch of a LAFT category

AKA: Yoneda always knock twice

Def A cocomplete category is LAFT if every cocontinuous functor $\mathcal{E} \longrightarrow D$ is a left adjoint.

Example
- Cocomplete categories with a dense generator are LAFT
- Top is LAFT
- everything you can think of is LAFT.

Prop Let K be LAFT. Then,

$$K \simeq \mathrm{Cont}(K^{op}, \mathrm{Set}).$$

But this is telling us that K is sketched by the sketch $(K^{op}, \text{all limits}, \emptyset)$.

So K^{op} is a <u>tautological axiomatization</u> of K.

This "tautological remark" is the "peak in generality" of the theory of sketches, which shows that everything is "virtually sketchable". Of course, properties of the sketch influence properties of its models.

Also keep in mind that the sketch for Top

$$(\mathrm{Top}^{op}, \text{all lims}, \emptyset)$$

is very illegitimate from a size point of view.